On the dynamics of trap models in $\mathbb{Z}^d$

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Abstract

We consider trap models in $\mathbb{Z}^d$. These are stochastic processes in a random environment as follows. The environment is given by a family $\tau = (\tau_x\,,\, x \in \mathbb{Z}^d)$ of positive iid random variables in the basin of attraction of an α-stable law, $0 < \alpha < 1$. Given τ, our process is a continuous time Markov pure jump process, whose jump chain is a in principle generic random walk in $\mathbb{Z}^d$, $d \geq 1$, independent of τ, and τ represents the holding time averages of the continuous time process. We may think of the sites of $\mathbb{Z}^d$ as traps, and of τ_x as the depth of trap x. We are interested in the *trap process*, namely the process that associates to time t the depth of the currently visited trap. Our first result is the convergence of the law of that process under a suitable scaling. The limit process is given by the jumps of a certain α-stable subordinator at the inverse of another α-stable subordinator, correlated with the first subordinator. For that result, the requirements on the underlying random walk are a) the validity of a law of large numbers for its range, and b) the slow variation at infinity of the tail of the distribution of its time of return to the origin: this includes all transient random walks as well as all random walks in $d \geq 2$, and also many one dimensional random walks. We then derive *aging results* for our process, namely scaling limits for some two-time correlation functions of the process. The scaling limit result mentioned above is an integrated result with respect to the environment. Under an additional condition on the size of the intersection of the ranges of two independent copies of the underlying random walk, roughly saying that it is relatively small compared with the range, we derive a stronger scaling limit result, roughly stating that it holds in probability with respect to the environment. With that additional condition, we also strengthen the aging results, from the integrated version mentioned above, to convergence in probability with respect to the environment.

Keywords and Phrases: trap models, random walks, scaling limit, aging, subordinators, random environment

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1 Introduction

We begin with a more precise definition of random walks among random traps. These are constructed through the following two-step procedure. We first choose a probability measure

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on $\mathbb{Z}^d \setminus \{0\}$, say μ, and let $\tau = (\tau_x\,,\, x \in \mathbb{Z}^d)$ be a collection of positive real numbers attached to the points of $\mathbb{Z}^d$. The *random walk in the trap environment* τ is then the continuous time Markov process with values in $\mathbb{Z}^d$ that starts at the origin and has generator

$$\mathcal{L}^\tau f(x) = \frac{1}{\tau(x)} \sum_y (f(x) - f(y))\mu(y - x)\,, \tag{1.1}$$

so that when sitting at point $x \in \mathbb{Z}^d$, the process waits for an exponentially distributed time of mean τ_x and then jumps to point $x + y$ where y is sampled from the distribution μ. This procedure is then iterated with independent hopping times and jumps. Next we choose τ at random such that the $(\tau_x\,,\, x \in \mathbb{Z}^d)$ are independent identically distributed random variables whose common law belongs to the domain of attraction of a stable law of index $\alpha \in (0, 1)$. The model thus defined is therefore an example of a random walk in a random environment. We denote with $\mathbb{P}$ the probability thus defined. More precisely, $\mathbb{P}$ is a probability measure on the product space $\Omega \times \mathcal{D}([0, \infty), \mathbb{Z}^d)$ where $\Omega = (0, \infty)^{\mathbb{Z}^d}$ is the space of *trap environments* and $\mathcal{D}([0, \infty), \mathbb{Z}^d)$ is the space of cad-lag trajectories from $[0, \infty)$ to $\mathbb{Z}^d$. The first marginal of $\mathbb{P}$ is of the form $Q^{\mathbb{Z}^d}$, where Q belongs to the domain of attraction of a stable law of index $\alpha \in (0, 1)$, and the conditional law of the second marginal given τ is the law of the random walk in the trap environment τ. We use the notation $(\mathcal{X}_t\,,\, t \geq 0)$ for the canonical projections defined on $\mathcal{D}([0, \infty), \mathbb{Z}^d)$ that give the position of the random walker at times $(t \geq 0)$.

Remark 1 *It is not difficult to see that if we choose Q with compact support in $(0, +\infty)$ then the behaviour of the random walk with traps is very similar to the random walk without traps. For instance, if μ is symmetric with finite support, one finds that the scaling limit of $\mathcal{X}$ under diffusive scaling is a Brownian motion. Fluctuations of the environment only affect the value of the effective diffusivity. In order to observe stronger slowing down effects, in particular aging, one has to choose heavy tailed τ's as we do here.*

This process is an example of a *trap model* in the spirit of J-P. Bouchaud. One important aspect of it is the lack of dependence of μ on τ. (A class of models where there is such a dependence, known as *asymmetric trap models*, have also been considered in the physics and mathematics literature. See below. Unless explicitely mentioned, we do not discuss these models here.) Such processes were initially introduced in the context of statistical mechanics as toy models for spin glasses and in order to illustrate the phenomenon of *aging*, see [12], [13] or [14] for instance. In usual models of spin glasses, the Hamiltonian is a random Gaussian field of large variance. At low temperature, it is natural to guess that the main contributions to the dynamics come from states of low energy. As the statistical properties of extremes of log Gaussian fields, the Gibbs factors in this context, are described by random variables with polynomial tail, the choice of a law in the basin of attraction of a stable law for τ, which plays a similar role in the simplified model, is also natural. Note that the parameter α can then be interpreted as the temperature, see [2] and Subsection 3.2 in [18].

The aging property refers to the following phenomenon: as time increases, the process visits a larger and larger part of its state space and therefore increases its probability to find a location x where τ_x is large. Since the time the process stays at location x before jumping off is of order τ_x, some slow down effect might take place. One way to measure how much the process is slow is to compute quantities of the form

$$\Pi(s,t) = \mathbb{P}(\mathcal{X}_r = \mathcal{X}_t\,,\ r \in [t, t+s])\,, \tag{1.2}$$

which are generally called *aging functions* in this context. The Markov property implies that

$$\Pi(s,t) = \mathbb{E}(e^{-s/\tau_{\mathcal{X}_t}})\,, \tag{1.3}$$

and thus we observe that a non trivial limit for $\Pi(s,t)$ as $s, t \to \infty$, with s and t related in a given way, implies that, at large time t, $\tau_{\mathcal{X}_t}$ should be of order s, so that the (order of the) 'age' of the process can actually be approximately read from its position at large times. Thus, in order to describe aging, we are led to considering the asymptotics of the *energy* (or *trap*) *process* $(\mathcal{E}_t = \tau_{\mathcal{X}_t}\,,\ t \geq 0)$.

The first computations of J-P. Bouchaud and D. Dean in [12] and [13] consisted in describing the asymptotics of trapped random walks on a large complete graph and in some appropriate scaling. Since then, the subject has developped into a rich mathematical theory. Mathematical papers treating the model in the complete graph include [15] and [19]. Although one motivation is certainly to understand the physicists' claims and prove aging for as realistic as possible models of spin glasses, see [4], [5] and more recently [3], it also turns out that trapping and aging effects also play a role in models without any connection to spin glass theory such as random walks with random conductances or random walks on Galton-Watson trees, see [1], [10]. The main strategy used in these papers has a strong potential theoretic flavour: for a given realisation of the trap environment τ, one tries to identify, among the different points x with large τ_x which will be hit by the random walk. We refer to [8] for a presentation of this point of view in an abstract setting. One advantage of this approach is that is does not seem to require the state space to have many symmetries. It provides strong forms of aging properties that are valid for a given realisation of the traps. On the other hand this machinery is often quite heavy to use.

As far as trap models on $\mathbb{Z}^d$ are concerned, excluding the asymmetric case (where μ depends on τ in a specific way, as mentioned above; see [1], [6], [24]), only the case of the simple symmetric random walk was investigated so far. It corresponds to μ being the uniform law on the nearest neighbors of the origin. Then the paths of the process $\mathcal{X}$, i.e. the sequence of the different points visited by $\mathcal{X}$, is a symmetric nearest neighbor random walk on $\mathbb{Z}^d$. The speed at which the process $\mathcal{X}$ moves i.e. the different hopping times at the successive locations are given by the environment τ. The one-dimensional case happens to be special: due to the strong recurrence properties of the simple symmetric random walk on $\mathbb{Z}$, the process gets localized. This localization effect, aging properties and scaling limits are precisely described in [17]. The scaling limit is a singular diffusion now known under the name of *FIN*. In higher dimension $d \geq 2$, the scaling limit is known to be the so called *Fractional Kinetics* process: d-dimensional Brownian motion time changed by the inverse of an independent stable subordinator as proved in [16], [9] and [7], the strategy being similar to [8]. Besides a bunch of estimates on the Green kernel of simple symmetric random walk in $\mathbb{Z}^d$, the proof in the $d = 2$ case involves rather sophisticated renormalization technics. This also follows in $d \geq 5$ as a particular case of results in [24], where a different approach is developed.

What do we do here? A first motivation of this paper is to derive aging properties for a more general class of random walks than the nearest neighbors case, in the form of an appropriate scaling limit of the energy process, as suggested by our discussion above; see Theorem 5 in

Section 2 below. In doing so we hope to clarify which properties of the random walk are truly relevant for aging. Observe in particular that the usual recurrence versus transience dichotomy does not apply here (as we can already conclude from the results of [7] for the simple symmetric case). As an outcome, we obtain a new proof of aging that applies to any genuinely d dimensional random walk for $d \geq 2$. This proof is more conceptual than the approach previously used by other authors. Indeed we need to know very little about specific estimates for the transition probabilities or Green kernels. We also completely avoid the renormalization step, even in the $d = 2$ case. It should also be mentioned that Theorem 5 is an annealed result with respect to the environment, as opposed to the analogue quenched result of [7]. In particular it applies in situations where $\mathcal{X}$ does not have a quenched scaling limit, see Remark 26 in Section 6 below. It can however be strengthened with little more effort, say, half way towards a quenched result, under an additional condition on the intersections of the ranges of independent copies of our process (see Theorem 25 in Section 6 below).

Observe that in our general setting it does not really make sense to look for scaling limits of the process $\mathcal{X}$ itself. Indeed the underlying random walk (with increments distributed according to Q) may not have a scaling limit. We choose then to focus on the energy process $\mathcal{E}$, which is natural to do in the aging context, as we discuss next. From the expression of the aging function $\Pi(s,t)$, it would be sufficient to know the limiting law of $\mathcal{E}_t/s$ to derive an aging property. We actually provide a much more complete answer by describing the scaling limit of the full process $\mathcal{E}$, thus establishing a fuller aging picture. This scaling limit is expressed as the value of the jump of some subordinator computed at the inverse of another subordinator. Interestingly, this scaling limit is universal, even if the scale on which the process $\mathcal{E}$ lives depends on the random walk (and in particular is linear if and only if the random walk is transient).

The topology under which we are able to establish Theorem 5 is quite weak, due to the nature of the energy process. Obtaining a scaling limit result for aging functions like (1.2) requires more work, done in Section 5 (for (1.2) and two other examples) under the original conditions, and in Subsection 6.1 with the additional condition of Section 6; see Theorems 18 and 29.

The remainder of this paper is organized as follows. In Section 2 we have a detailed presentation of the model, assumptions and one result (annealed scaling limit of $\mathcal{E}$), with some more discussion. In Section 3, we discuss some preliminary results on random walks that are used subsequently. In Section 4, we prove the annealed result just mentioned, and in Sections 5 and 6 we state and prove our further scaling limit results for some aging functions, and in a stronger than annealed sense, as discussed above.

2 Model and results

As in the introduction, let μ be a probability measure on $\mathbb{Z}^d \setminus \{0\}$, and let $\tau = (\tau_x \, , \, x \in \mathbb{Z}^d)$ be a collection of positive numbers attached to the points of $\mathbb{Z}^d$ chosen as follows. Let Q be a probability measure on $(0, \infty)$ that belongs to the domain of attraction of a stable law of index $\alpha \in (0, 1)$. In other words we assume that

$$Q(t, \infty) = \ell(t) \, t^{-\alpha} \tag{2.1}$$

where ℓ is a slowly varying function at infinity. We choose for

$$\tau = \{\tau_x \, , \, x \in \mathbb{Z}^d\} \tag{2.2}$$

a family of independent random variables with law Q. More precisely we endow the product space $\Omega = (0, \infty)^{\mathbb{Z}^d}$ with the law $\mathcal{Q} = Q^{\mathbb{Z}^d}$.

We consider the Markov generator

$$\mathcal{L}^\tau f(x) = \frac{1}{\tau(x)} \sum_y (f(x) - f(y))\mu(y - x) \, . \tag{2.3}$$

Let P^τ_x be the law of the Markov process $\mathcal{X}$ generated by $\mathcal{L}^\tau$ and started at x on path space $\mathcal{D}([0, \infty), \mathbb{Z}^d)$. We recall that $(\mathcal{X}_t \, , \, t \geq 0)$ denotes the canonical projections on $\mathcal{D}([0, \infty), \mathbb{Z}^d)$. We define the energy process:

$$\mathcal{E} = (\mathcal{E}_t = \tau_{\mathcal{X}_t} \, , \, t \geq 0) \, . \tag{2.4}$$

The so-called *annealed law* of the process $\mathcal{X}$ is the semidirect product measure on $\Omega \times \mathcal{D}([0, \infty), \mathbb{Z}^d)$ defined by

$$\mathbb{P}(A \times B) = \int_A d\mathcal{Q}(\tau) P^\tau_0(B), \tag{2.5}$$

where A and B are measurable subsets of Ω and $\mathcal{D}([0, \infty), \mathbb{Z}^d)$ respectively.

In order to state our assumptions we introduce an auxiliary random walk: let $\xi_1, \xi_2, \dots$ be iid $\mathbb{Z}^d$-valued random vectors with distribution μ and define

$$X_0 = 0, \quad X_n = \sum_{i=1}^n \xi_i, \quad n \geq 1 \, . \tag{2.6}$$

Also define the *range* of X (*up to time* n),

$$\mathcal{R}_n = \mathcal{R}_n(X) = \{z \in \mathbb{Z}^d : \, X_i = z \text{ for some } i \leq n\}, \quad n \geq 0 \tag{2.7}$$

and make

$$R_n = |\mathcal{R}_n| \text{ and } \rho_n = \mathbb{E}(R_n). \tag{2.8}$$

Also define for $n \geq 1$

$$r_n = \mathbb{P}(X_1 \neq 0, \dots, X_n \neq 0) \, . \tag{2.9}$$

We will at times think of (r_n) as a function.

Our first result requires the following assumptions.

Assumption A (law of large numbers for the range):

$$\lim_{n\to\infty} \frac{R_n}{\rho_n} = 1 \text{ in probability.} \tag{2.10}$$

Assumption B (slow variation of r): $r : \mathbb{N} \to [0, 1]$ given in (2.9) above is slowly varying at infinity.

Remark 2 *All transient random walks in $\mathbb{Z}^d$, $d \geq 1$, including all random walks in $d \geq 3$, obviously satisfy Assumptions A and B. But all planar random walks [20, 21], and 1-dimensional β-stable random walks with $\beta \leq 1$ [23] also satisfy Assumptions A and B.*

Before stating the result, we describe the form of the scaling limit of $\mathcal{E}$: we introduce an α-stable subordinator $\Upsilon = (\Upsilon_t)_{t\geq 0}$ and a family of independent mean 1 exponential random variables $\{T_t;\, t \geq 0\}$, and let

$$V_t = \int_0^t T_s \, d\Upsilon_s \tag{2.11}$$

and $W = V^{-1}$ be the right continuous inverse of V. Let finally

$$Z_t = \Upsilon(W_t) - \Upsilon(W_t-), \quad t \geq 0. \tag{2.12}$$

Remark 3 *Note that V is a stable subordinator equidistributed with Υ. We may regard V and Z as processes in the random environment Υ. Indeed, given Υ, they are both Markovian. We may think of Υ as the scaling limit of the (relevant) environment of the trap model. See discussion below. The overall distribution of Z (integrated over the joint distribution of Υ and $\{T_t;\, t \geq 0\}$) makes it a self similar process of index 1.*

Remark 4 *Let $c > 0$ be the constant such that $\mathbb{E}(\exp\{-\lambda\Upsilon_1\}) = \exp\{-c\lambda^\alpha\}$. One readily checks that the distribution of Z does not depend on that constant. For convenience, we will take it so that (4.11) below is satisfied.*

We are now ready to state our convergence result, but first some notation. For $\varepsilon > 0$, $t \geq 0$, let

$$\mathcal{E}_t^{(\varepsilon)} = a_\varepsilon \mathcal{E}_{\varepsilon^{-1}t}, \tag{2.13}$$

where a_ε is to be specified below, and denote $\mathcal{E}^{(\varepsilon)} = (\mathcal{E}_t^{(\varepsilon)})$ and $Z = (Z_t)$. Let D, D_T denote the spaces of càdlàg real functions defined on $[0,\infty)$, $[0,T]$ respectively. Let d_T denote the L_1 distance in D_T, and $d = \sum_{n=1}^\infty 2^{-n}(d_n \wedge 1)$.

Theorem 5 *There exists $(a_\varepsilon)_{\varepsilon>0}$ such that as $\varepsilon \searrow 0$*

$$\mathcal{E}^{(\varepsilon)} \to Z \tag{2.14}$$

in distribution on (D, d).

In the remainder of this section we roughly explain the magnitude of the scaling in the above result, the form of the limit, as well as the issues involving the arguments for the scaling limits of aging functions and the strengthening of the annealed aspect of Theorem 5.

Let $(x_j\,,\, j \geq 0)$ be the successive points visited by a trajectory $\mathcal{X}$. Under P_x^τ, we observe that the law of $x.$ is the same as the law of the random walk X in (2.6), and thus Assumptions A and B above hold for $x.$. In particular the law of $x.$ under P_x^τ does not depend on τ. Let us further define t_j to be the time spent by $\mathcal{X}$ at point x_j before jumping to point x_{j+1} divided by τ_{x_j} so that, under P_x^τ, the random variables $(t_j\,,\, j \geq 0)$ are independent, mean 1 exponential variables. Also note that the collections $(t_j\,,\, j \geq 0)$ and $(x_j\,,\, j \geq 0)$ are independent of each other.

Conversely, one can retrieve the trajectory $\mathcal{X}$ from the data $t.$ and $x.$ in the following way: define the *clock process* by

$$c_n = \sum_{j=0}^{n} \tau_{x_j}\, t_j = \sum_{x\in\mathcal{R}_n} \tau_x\, \ell(x,n) = \sum_{i=1}^{R_n} \tau_{\tilde{x}_i}\, \ell(\tilde{x}_i, n), \tag{2.15}$$

where for $x \in \mathbb{Z}^d$, $\ell(x,n) = \sum_{j=0}^{n} \mathbf{1}(x_j = x)\, t_j$ is the occupation time of point x until time n and $\{\tilde{x}_i,\ i = 1, 2, \dots\}$ are the *distinct* sites of $(x_j\,,\ j \geq 0)$, in the same order as they appear in $(x_j\,,\ j \geq 0)$. Let $(i_t\,,\ t \geq 0)$ be the right continuous inverse of $c_\cdot$. Then $\mathcal{X}_t = x_{i_t}$, and $\mathcal{E}_t = \tau_{x_{i_t}}$.

Assumption A states that $R_n \sim \rho_n$. Let

$$s_n = \inf\{t \geq 0 : Q(t, \infty) \leq n^{-1}\}. \tag{2.16}$$

We recall that s_n gives the correct order of magnitude of a sum of n independent variables of law Q. Assumption A implies that in (2.15) we may disregard all terms of order $o(s_{\rho_n})$. Assumption B allows the conclusion, as will be established below, that among the $O(s_{\rho_n})$ sites of $\mathcal{R}_n$, say $\{\tilde{x}'_i,\ i = 1, 2, \dots\}$, the random variables $r_n\, \ell(\tilde{x}'_i, n)$, $i = 1, 2, \dots$, are approximately iid mean 1 exponentials. We then have that c_n equals in distribution approximately

$$\tilde{c}_n = r_n^{-1} \sum_{i=1}^{\rho_n} \tau_{\tilde{x}_i}\, \tilde{t}_i. \tag{2.17}$$

where $\tilde{t}_i$, $i = 1, 2, \dots$, are iid mean 1 exponentials. Now letting $\hat{c}_n = \sum_{i=1}^{\rho_n} \tau_{\tilde{x}_i}$, we find that

$$\mathcal{E}_t \approx \hat{c}_{\tilde{\imath}_t} - \hat{c}_{\tilde{\imath}_t - 1}, \tag{2.18}$$

the jump of $\hat{c}$ at the inverse $\tilde{\imath}$ of $\tilde{c}$, where $\approx$ means approximate equality in distribution. We thus see the forms of (2.11) and (2.12) emerging from those of (2.17) and (2.18), respectively. The time scale is that of the clock $c_n \sim \tilde{c}_n \sim \nu_n := v_n / r_n$, where $v_n = s_{\rho_n}$, and the "energy" scale is that of $\hat{c}_n \sim v_n$. So to times of order ε^{-1} correspond energies of order $v_{n(\varepsilon^{-1})}$, where $n(\cdot)$ is the inverse of ν. We thus take

$$a_\varepsilon = 1/v_{n(\varepsilon^{-1})}. \tag{2.19}$$

The weak convergence of $\hat{c}$, which may be regarded as the (relevant) environment for the trap model, to Υ closes the picture. See Remark 3 above.

A remark about the topology: with the disregarding of the small traps suggested above, the convergence would take place in the J_1/usual Skorohod topology, but when those traps are put back in, they mix up with the large ones, in such a way that the J_1 topology (and other more usual ones, like the M_1 topology) is too fine to handle, and so we need a rougher topology, like the one we use.

One technical aspect to consider in order to prove the scaling limit of an aging function like (1.3) is to show that for an arbitrary fixed time t, $\mathcal{E}_t^{(\varepsilon)}$ converges to Z_t as $\varepsilon \to 0$. This does not follow from Theorem 5, mainly due to the topological issue just discussed. We state and prove a convergence result for a fixed time, Lemma 21, in Section 5, before addressing the main issue.

We close with a brief discussion on the stronger than annealed version for Theorem 5. Roughly, the reason why Theorem 5 is an annealed result is that the convergence of scaled $\hat{c}$ to Υ is a weak one when $\{\tilde{x}_i\}$ is fixed. But of course, when we consider the distribution of, say, scaled $\hat{c}$ given τ, we integrate with respect to $\{\tilde{x}_i\}$, and the averaging that this involves could lead to a stronger result. A condition is required, however (annealed convergence is all we have in, say, the asymmetric simple one dimensional case). One is introduced in Section 6, saying roughly that independent realizations of the trajectory of X intersect little. With that additional condition, we state and prove stronger convergence results.

3 Preliminaries on random walks

In this section, we establish a few facts concerning discrete time random walks that follow from Assumptions A and B made in Section 2 above, as well as from other assumptions we will consider below. These results will be used later in the sections ahead.

Let $X = (X_n,\, n \geq 0)$ be the random walk introduced in Section 2 above, and define

$$u_n = \mathbb{P}(X_n = 0), \quad U_n = \sum_{i=0}^{n} u_i, \quad L_n = \sum_{i=0}^{n} \mathbf{1}\{X_i = 0\}, \quad n \geq 0. \tag{3.1}$$

L_n is the occupation time of the origin up to step n. We will also write L_x for a positive real x, and it means $L_{\lfloor x \rfloor}$, similarly for U_x and r_x.

Our first remark is that

$$\rho_n = \sum_{k=0}^{n} r_k, \tag{3.2}$$

with ρ and r defined in (2.8, 2.9) above. The formula is proved as follows (see [26] page 36): write that

$$R_n = \sum_{k=0}^{n} \phi_k\,,$$

where ϕ_k is the indicator function that R (defined in (2.7) above) increases by 1 at time k. That is

$$\phi_k = \mathbf{1}\{R_k = R_{k-1} + 1\} = \mathbf{1}\{X_k - X_{k-1} \neq 0; X_k - X_{k-2} \neq 0, \ldots, X_k \neq 0\}\,.$$

So

$$\mathbb{E}(\phi_k) = \mathbb{P}(\xi_k \neq 0; \xi_k + \xi_{k-1} \neq 0, \ldots, X_k \neq 0) = \mathbb{P}(X_1 \neq 0; X_2 \neq 0, \ldots, X_k \neq 0) = r_k\,.$$

It then follows from (3.2) and Assumption B that

$$\lim_{n\to\infty} \frac{\rho_n}{n r_n} = 1\,. \tag{3.3}$$

(This readily follows from the first displayed equation on page 55 of [25].)

Our second remark is the following result.

Lemma 6 *Under Assumption B, and provided $\lim_{n\to\infty} r_n = 0$ (that is, if X is recurrent), the law of $r_n L_n$ approximates a mean 1 exponential distribution as $n \to \infty$.*

Proof

Given $u \geq 0$, we have that

$$\{r_n L_n > u\} = \{\eta_1 + \ldots + \eta_k \leq n\}$$

where the η_j, $j \geq 1$, are the successive increments of return times to the origin by X, and $k = \lfloor u/r_n \rfloor + 1$.

Therefore

$$\mathbb{P}(r_n L_n > u) = \mathbb{P}(\eta_1 + ... + \eta_k \le n) = \mathbb{P}(\bar{\eta}_n \le 1), \tag{3.4}$$

where $\bar{\eta}_n = (\eta_1 + ... + \eta_k)/n$.

A straightforward computation of the Laplace transform of $\bar{\eta}_n$ yields

$$\mathbb{E}(e^{-\lambda\bar{\eta}_n}) = \{\mathbb{E}(e^{-\frac{\lambda}{n}\eta_1})\}^k = \left\{1 - \frac{\lambda}{n}\int_0^\infty r(x)\, e^{-\frac{\lambda}{n}x}\, dx\right\}^k = \left\{1 - \int_0^\infty r(yn/\lambda)\, e^{-y}\, dy\right\}^k, \tag{3.5}$$

where $r(x) = \mathbb{P}(\eta_1 > x) = r_{\lfloor x \rfloor}$.

Assumption B and Theorems 2.6 and 2.7 of [25] imply that the integral on the right hand side of (3.5) is asymptotic to r_n as $n \to \infty$. This and the form of k imply that

$$\lim_{n\to\infty} \mathbb{E}(e^{-\lambda\bar{\eta}_n}) = e^{-u} \tag{3.6}$$

for all $\lambda > 0$, and this implies that the law of $\bar{\eta}_n$ converges to that of a(n extended) random variable which takes the value 0 with probability e^{-u}, and the value ∞ with the complementary probability. It follows that

$$\lim_{n\to\infty} \mathbb{P}(r_n L_n > u) = e^{-u} \tag{3.7}$$

for every $u > 0$. $\square$

Corollary 7 *Under Assumption B, we have that*

$$\lim_{n\to\infty} r_n U_n = 1. \tag{3.8}$$

Proof

In the transient case, this follows from $r_n \to r_\infty > 0$, $L_n \to L_\infty$, a Geometric random variable with mean r_∞^{-1}, and monotone convergence.

In the recurrent case, from the first equality in (3.4), we find that

$$\mathbb{P}(r_n L_n > u) \le \mathbb{P}\left(\max_{1\le j\le k} \eta_j \le n\right) = (1 - r_n)^k. \tag{3.9}$$

From the form of k, and the fact that $\lim_{n\to\infty}(1 - r_n)^{1/r_n} = e^{-1}$, we find that for every $c > e^{-1}$ and large enough n, the right hand side of (3.8) is dominated by c^u for all large enough n. Dominated convergence now yields

$$r_n U_n = \int_0^\infty \mathbb{P}(r_n L_n > u)\, du \to \int_0^\infty e^{-u}\, du = 1\,. \tag{3.10}$$

as $n \to \infty$, since, from (3.1), $r_n U_n = r_n \mathbb{E}(L_n) = \mathbb{E}(r_n L_n)$. $\square$

Another corollary of Lemma 6 is as follows. It concerns the random variable $\ell(0, n)$, whose generic definition was given right below (2.15).

Corollary 8 *Under Assumption B, we have that $r_n \ell(0, n)$ converges weakly to a mean 1 exponential distribution.*

Proof

In the recurrent case, it follows immediately from Lemma 6 and the law of large numbers, once we observe that

$$\ell(0,n)=\sum_{i=1}^{L_n} t'_j \tag{3.11}$$

in distribution, where t'_j, $j\geq 1$, are iid mean 1 exponential random variables independent of L_n.

In the transient case, L_n converges as $n\to\infty$ to a geometrically distributed random variable, say L_∞. Also $\lim_{n\to\infty} r_n = r_\infty > 0$, and one readily checks that $r_\infty \sum_{i=1}^{L_\infty} t'_j$ is a mean 1 exponential random variable. □

Lemma 9 *Under Assumption B, we have that for every* $0<a<b<\infty$

$$L_{bn}-L_{an}\to 0 \tag{3.12}$$

in probability as $n\to\infty$*.*

Remark 10 *Since* $L_{bn}-L_{an}$ *is an integer, we have that the probability of no return to* 0 *of* X *during* $[an,bn]$ *goes to 1 as* $n\to\infty$ *for every fixed* $0<a<b<\infty$*.*

Proof

Let $N=N_n(a,b)$ denote the random variable on the left of (3.12). Then by the Markov property, one readily checks that the conditional distribution of $L_{(b+1)n}-L_{an}$ given $N\geq 1$ dominates the unconditional one of L_n. We conclude that

$$\begin{aligned} U_{(b+1)n}-U_{an} &= \mathbb{E}[L_{(b+1)n}-L_{an}]\geq \mathbb{E}[L_{(b+1)n}-L_{an};\, N\geq 1] \\ &\geq \mathbb{E}[L_n]\,\mathbb{P}(N\geq 1)=U_n\,\mathbb{P}(N\geq 1). \end{aligned} \tag{3.13}$$

We then have that $\mathbb{P}(N\geq 1)\leq (U_{(b+1)n}-U_{an})/U_n$ and the result follows from Assumption B and (3.8). □

4 Convergence

Let X and τ be as in the previous sections. Assumptions A and B are in force.

From now on we will adopt a particular construction of $\mathcal{X}$ and $\mathcal{E}$, as follows.

Let $T_0,T_1,T_2,\dots$ be a family of independent mean 1 exponential random variables. Consider the following random function $C:\mathbb{N}\to[0,\infty)$:

$$C_n=\sum_{i=0}^{n}\tau_{X_i}\,T_i,\quad n\geq 0, \tag{4.1}$$

and let I denote its right continuous inverse.

Now define for $t\geq 0$

$$Y_t=\tau_{X_{I_t}}. \tag{4.2}$$

Remark 11 *Note that* (X_{I_t}) *is a version of* $\mathcal{X}$ *and* $Y=(Y_t)$ *is a version of* $\mathcal{E}$*. Thus, making* $Y_t^{(\varepsilon)}=a_\varepsilon Y_{\varepsilon^{-1}t}$*,* $t\geq 0$*, we have that* $Y^{(\varepsilon)}=(Y_t^{(\varepsilon)})$ *is a version of* $\mathcal{E}^{(\varepsilon)}$*. Below, we will prove Theorem 5 with* $Y^{(\varepsilon)}$ *replacing* $\mathcal{E}^{(\varepsilon)}$*. (Actually, we will introduce and use yet other versions.)*

Proof of Theorem 5

Before going to the details, we give the main ideas and steps of the argument.

We first observe from (4.1) and (4.2) that only the values of the environment along the trajectory of X actually matter and, whenever the process X discovers a new point it had not visited so far, say x, then the value of τ_x is independent of the past and distributed according to Q, regardless of the value of x. Thus we may think of the environment as being constructed from a sequence of i.i.d. Q distributed random variables step by step, each time X visits a new point. This is expressed in the version of the clock process $\tilde{C}$ in (4.5). Note that we have to keep track of the different times at which a new point is discovered by X: this job is done by function φ in (4.7).

The next version of the clock process, namely $\hat{C}^{(\varepsilon)}$ in (4.15) is just a rescaling of $\tilde{C}$.

Next we choose a specific version of the environment coupled with the jumps of the subordinator Υ, see (4.18)-(4.21). This is convenient because then we can consider the realisation of Υ as fixed and prove almost sure convergence. The new clock process is now denoted with $\bar{C}^{(\varepsilon)}$ and the proof of Theorem 5 boils down to Lemma 12.

We begin the proof of Lemma 12 by introducing a threshold parameter δ in order to select a finite number of jumps of Υ that most contribute to the clock process: the *deep traps* $x_1^{(\varepsilon)}, x_2^{(\varepsilon)}, \ldots$. Lemma 12, after the introduction of the δ parameter becomes Lemma 15.

Observe that equation (4.34) gives the clock process $\bar{C}^{(\varepsilon,\delta)}$ as a finite sum (for fixed time t and a given realisation of Υ) of jumps of Υ multiplied by the independent occupation times $\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)},t)$. By Corollary 8, we already know that, for each i, the occupation time $\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)},t)$ is either close to 0 (if point $x_i^{(\varepsilon)}$ has not been visited yet) or close to exponential. The independence of the $\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)},t)$'s is a consequence of the fact that the main contribution of the successive visits to $x_i^{(\varepsilon)}$ occur before the hitting time of $x_{i+1}^{(\varepsilon)}$, see (4.38) and after.

At that point, we will have constructed a good approximation of the clock process C by a finite sum of increments of a subordinator multiplied by iid exponentials which is the key step in the proof of Theorem 5.

Now the details. We begin by considering

$$\sigma_0=0, \quad \sigma_n=\min\{i\geq\sigma_{n-1}:\, X_i\notin\{X_0,\ldots,X_{i-1}\}\}, \quad n\geq 1, \tag{4.3}$$

and

$$\tilde{X}_n=X_{\sigma_n}, \quad n\geq 0, \tag{4.4}$$

and then we introduce the independent families of iid random variables $\{\tilde{\tau}_n,\, n\geq 0\}$ and $\{\tilde{T}_i^{(j)},\, j\geq 0, i\geq 1\}$, with $\tilde{\tau}_0\overset{d}{=}\tau_0$ and $\tilde{T}_1^{(0)}\overset{d}{=}T_0$ (see paragraph of (2.2) above), where $\overset{d}{=}$ means "equal in distribution".

If we now define $\tilde{C}:\mathbb{N}\to[0,\infty)$ such that

$$\tilde{C}_n=\sum_{j=0}^{\infty}\tilde{\tau}_j\sum_{i=1}^{L(\tilde{X}_j,n)}\tilde{T}_i^{(j)}, \quad n\geq 0, \tag{4.5}$$

where $L(x,n)=\sum_{i=0}^{n} 1\{X_i=x\}$ is the number of visits of X to $x\in\mathbb{Z}^d$ up to time $n\geq 0$, and $\sum_{i=1}^{0}\tilde{T}_i^{(j)}=0$ for any j. Then $\tilde{C}$ and C have the same distribution. To see this latter point, it may help to first notice that, given τ, $(C_n)_{n\geq 0}$ is equally distributed with

$$\sum_{j=0}^{\infty}\tau_{\tilde{X}_j}\sum_{i=1}^{L(\tilde{X}_j,n)}\tilde{T}_i^{(j)},\quad n\geq 0, \tag{4.6}$$

and then that, for almost every realization of X, $\{\tau_{\tilde{X}_n},\, n\geq 0\}$ and $\{\tilde{\tau}_n,\, n\geq 0\}$ are equally distributed.

Let now $\tilde{I}$ denote the right continuous inverse of $\tilde{C}$, and consider the map $\varphi:\mathbb{N}\to\mathbb{N}$ such that

$$\varphi(n)=m\quad\text{iff}\quad \tilde{X}_m=X_n. \tag{4.7}$$

Defining

$$\tilde{Y}_t=\tilde{\tau}_{\varphi(\tilde{I}_t)},\; t\geq 0, \tag{4.8}$$

we have that

$$(Y_t)_{t\geq 0}\quad\text{and}\quad(\tilde{Y}_t)_{t\geq 0}\quad\text{have the same distribution.} \tag{4.9}$$

We introduce now

$$\tilde{S}_n=\sum_{j=0}^{n}\tilde{\tau}_j,\quad n\geq 0., \tag{4.10}$$

and let s_n be as in (2.16) so that

$$(s_n^{-1}\tilde{S}_{\lfloor nt\rfloor})_{t\geq 0}\to\Upsilon \tag{4.11}$$

in distribution as $n\to\infty$, with Υ introduced above (see paragraph of (2.11) and Remark 4).

Let us again consider

$$v_n=s_{\rho_n}\quad\text{and}\quad \nu_n=\frac{v_n}{r_n} \tag{4.12}$$

(these quantities were introduced right above (2.19)).

We remark (again) that ν_n is the natural scaling for $\tilde{C}_n$ (have already done that right above (2.19)), but we want to consider a continuous scaling parameter ε (which we eventually let go to 0). For that reason, we consider the right continuous inverse of ν_n, denoted $n(\cdot)$. We then have that

$$\varepsilon\nu_{n(\varepsilon^{-1})}\to 1\quad\text{as}\quad\varepsilon\to 0, \tag{4.13}$$

and for any $t>0$ fixed

$$\frac{n(\varepsilon^{-1}t)}{n(\varepsilon^{-1})}\to t^{\alpha}\quad\text{as}\quad\varepsilon\to 0. \tag{4.14}$$

We now consider

$$\hat{C}_t^{(\varepsilon)}=\varepsilon\sum_{j=0}^{\infty}\tilde{\tau}_j\sum_{i=1}^{L(\tilde{X}_j,n(\varepsilon^{-1}t))}\tilde{T}_i^{(j)},\quad t\geq 0, \tag{4.15}$$

and let $\hat{I}^{(\varepsilon)}$ denote its right continuous inverse, and then define

$$\hat{Y}_t^{(\varepsilon)}=a_\varepsilon\,\tilde{\tau}_{\varphi(n(\varepsilon^{-1}\hat{I}_t^{(\varepsilon)}))}, \tag{4.16}$$

where a_ε was defined in (2.19) above. We then have that

$$(\hat{Y}_t^{(\varepsilon)}) = (a_\varepsilon \tilde{Y}_{\varepsilon^{-1}t}) = Y^{(\varepsilon)} \tag{4.17}$$

in distribution.

We change now $(\tilde{\tau}_j)$ to a more convenient version. Let $\tilde{\varepsilon} = \rho_{n(\varepsilon^{-1})}^{-1}$. For $x \in \tilde{\varepsilon}\,\mathbb{N}$, let

$$\tau_x^{(\varepsilon)} = v_{n(\varepsilon^{-1})}\, g_\varepsilon(\Upsilon_{x+\tilde{\varepsilon}} - \Upsilon_x) \tag{4.18}$$

$$g_\varepsilon(y) = \frac{1}{v_{n(\varepsilon^{-1})}}\, G^{-1}(\tilde{\varepsilon}^{-1/\alpha} y),\ y \geq 0, \tag{4.19}$$

and G is defined by

$$\mathbb{P}(\Upsilon_1 > G(y)) = \mathbb{P}(\tau_0 > y),\ y \geq 0. \tag{4.20}$$

Then $\{\tau_x^{(\varepsilon)},\ x \in \tilde{\varepsilon}\,\mathbb{N}\}$ is equidistributed with $\{\tilde{\tau}_j,\ j \in \mathbb{N}\}$, and it follows that $\{\hat{C}_t^{(\varepsilon)},\ t \geq 0\}$ is equidistributed with

$$\bar{C}_t^{(\varepsilon)} = \sum_{x \in \tilde{\varepsilon}\mathbb{N}} g_\varepsilon(\Upsilon_{x+\tilde{\varepsilon}} - \Upsilon_x)\ \left\{ \bar{r}_{n(\varepsilon^{-1})} \sum_{i=1}^{L\left(\tilde{X}_{\tilde{\varepsilon}^{-1}x},\, n(\varepsilon^{-1}t)\right)} T_i^{(\tilde{\varepsilon}^{-1}x)} \right\}, \quad t \geq 0, \tag{4.21}$$

where

$$\bar{r}_{n(\varepsilon^{-1})} = \varepsilon \nu_{n(\varepsilon^{-1})}\, r_{n(\varepsilon^{-1})} = \varepsilon\, v_{n(\varepsilon^{-1})}. \tag{4.22}$$

(Note that by (4.13) we have that

$$\bar{r}_{n(\varepsilon^{-1})} \sim r_{n(\varepsilon^{-1})} \tag{4.23}$$

as $\varepsilon \to 0$.)

Let us denote the expression within curly brackets in (4.21) as $\bar{T}^{(\varepsilon)}(x,t)$. We remark, using the Markov property, that its expectation is bounded above by

$$\bar{r}_{n(\varepsilon^{-1})}\, \mathbb{E}(L(0, n(\varepsilon^{-1}t))) = \bar{r}_{n(\varepsilon^{-1})}\, U_{n(\varepsilon^{-1}t)} \to 1 \quad \text{as} \quad \varepsilon \to 0, \tag{4.24}$$

for every $t > 0$, by Corollary 7 and (4.23).

Let now $\bar{I}^{(\varepsilon)} : \mathbb{R}^+ \to \tilde{\varepsilon}\mathbb{N}$ denote the right continuous inverse of $\bar{C}^{(\varepsilon)}$, and make

$$\bar{Y}_t^{(\varepsilon)} = a_\varepsilon\, \tau^{(\varepsilon)}_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} = g_\varepsilon\left(\Upsilon_{\tilde{\varepsilon}\left(\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)+1\right)} - \Upsilon_{\tilde{\varepsilon}\varphi\left(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon)})\right)} \right). \tag{4.25}$$

We remark now that $(\bar{Y}_t^{(\varepsilon)})_{t\geq 0}$ and $(\hat{Y}_t^{(\varepsilon)})_{t\geq 0}$ have the same distribution. Together with (4.9) and (4.13), this means that to prove Theorem 5, it suffices to establish Lemma 12 below instead. $\square$

Lemma 12 *For almost every* Υ

$$\bar{Y}^{(\varepsilon)} \to Z \tag{4.26}$$

as $\varepsilon \searrow 0$ *in distribution on* (D, d).

Remark 13 *The distribution referred to in the above statement is the joint one of* (X_n) *and* $\{\tilde{T}_i^{(j)}\}$ *(with* Υ *fixed).*

Proof

It will be implicit (and sometimes explicit) in the claims made below that they hold for a.e.-Υ. We start by defining the set of *traps*. Let $\mu(x) = \Upsilon(x) - \Upsilon(x-)$ and fix $\delta > 0$ arbitrarily. Consider

$$\mathfrak{T}_\delta = \{x \geq 0 : \mu(x) > \delta\} = \{x_1 < x_2 < \ldots\}, \tag{4.27}$$

and let $x_i^{(\varepsilon)} = \tilde{\varepsilon}\lfloor\tilde{\varepsilon}^{-1}x_i\rfloor$, $i \geq 1$, and

$$\mathfrak{T}_\delta^{(\varepsilon)} = \{x_1^{(\varepsilon)}, x_2^{(\varepsilon)}, \ldots\}, \tag{4.28}$$

and make $\mu^{(\varepsilon)}(x) = g_\varepsilon(\Upsilon_{x+\tilde{\varepsilon}} - \Upsilon_x)$. Proposition 3.1 of [17] implies that for a.e.-Υ

$$\mu^{(\varepsilon)}(x_i^{(\varepsilon)}) \to \mu(x_i) \quad \text{as} \quad \varepsilon \to 0, \tag{4.29}$$

for every $i \geq 1$. (With the above notation, we may write

$$\bar{C}_t^{(\varepsilon)} = \sum_{x \in \tilde{\varepsilon}\mathbb{N}} \mu^{(\varepsilon)}(x)\, \bar{T}^{(\varepsilon)}(x,t).) \tag{4.30}$$

We then have by (4.24,4.29) that for every $K \geq 1$

$$\left|\sum_{i=1}^K \mu(x_i)\, \bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, t) - \sum_{i=1}^K \mu^{(\varepsilon)}(x_i^{(\varepsilon)})\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, t)\right| \leq \sum_{i=1}^K \left|\mu(x_i) - \mu^{(\varepsilon)}(x_i^{(\varepsilon)})\right| \bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, t) \to 0 \tag{4.31}$$

in probability as $\varepsilon \to 0$. We also claim that

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sum_{x \in \tilde{\varepsilon}\mathbb{N} \cap (\mathfrak{T}_\delta^{(\varepsilon)})^c} \mu^{(\varepsilon)}(x)\bar{T}^{(\varepsilon)}(x,t) = 0 \tag{4.32}$$

in probability for every $t \geq 0$. To justify that, we first observe that since $L(\tilde{X}_j, n(\varepsilon^{-1}t)) = 0$ if $j > R_{n(\varepsilon^{-1}t)}$, then by Assumptions A and B and (4.14), we can restrict the first sum in (4.32) on $x \leq t^\alpha + 1$; by (4.24) the expectation of the restricted sum is bounded above by constant times

$$\sum_{x \in \tilde{\varepsilon}\mathbb{N} \cap (\mathfrak{T}_\delta^{(\varepsilon)})^c \cap [0, t^\alpha + 1]} \mu^{(\varepsilon)}(x). \tag{4.33}$$

Now, as argued in [17] , we have that the $\lim_{\delta \to 0} \limsup_{\varepsilon \to 0}$ of (4.33) vanishes (see paragraphs of (3.25-3.28) in that reference), and the claim follows.

Let now

$$\bar{V}_t^{(\delta)} = \sum_{\substack{i=1,2,\ldots \\ x_i \leq t^\alpha}} \mu(x_i)\, T_{x_i}, \quad \bar{C}_t^{(\varepsilon,\delta)} = \sum_{i \geq 1} \mu^{(\varepsilon)}(x_i^{(\varepsilon)})\, \bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, t), \tag{4.34}$$

where an empty sum equals 0, and consider the right continuous inverses of $\bar{V}^{(\delta)}$ and $\bar{C}^{(\varepsilon,\delta)}$, $\bar{W}^{(\delta)}$ and $\bar{I}^{(\varepsilon,\delta)}$, respectively, and let

$$\bar{Z}_t^{(\delta)} = \mu(\bar{W}_t^{(\delta)}), \quad \bar{Y}_t^{(\varepsilon,\delta)} = \mu^{(\varepsilon)}\left[\tilde{\varepsilon}\,\varphi\Big(n(\varepsilon^{-1}\bar{I}_t^{(\varepsilon,\delta)})\Big)\right]. \tag{4.35}$$

Remark 14 *We may readily check that*

$$(\bar{V}_t^{(\delta)}) \to (V_t) \tag{4.36}$$

almost surely on (D, J_1) *as* $\delta \to 0$, *where* J_1 *is the usual Skorohod metric on* D.

Lemma 15 *Let* $\delta > 0$ *be fixed. Then as* $\varepsilon \searrow 0$

$$(\bar{Y}_t^{(\varepsilon,\delta)}) \to (\bar{Z}_t^{(\delta)}) \tag{4.37}$$

in distribution on (D, J_1).

Proof

We use the fact that $L(\tilde{X}_j, n(\varepsilon^{-1}t)) = 0$ if $j > R_{n(\varepsilon^{-1}t)}$ and Assumptions A and B and (4.14) to conclude that for ε small enough the second sum in (4.34) vanishes if $t^\alpha < x_1$, and is restricted to the k first terms if $x_k < t^\alpha < x_{k+1}$. We now introduce $\zeta_i^{(\varepsilon)}$, the hitting time by X of $\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}$, and observe that

$$L\Big(\tilde{X}_{\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}}, \zeta_{i+1}^{(\varepsilon)}\Big),\ i = 1, 2, \ldots \tag{4.38}$$

are independent. From the above we conclude that, given arbitrary t_j, $j = 1, 2, 3, 4$, such that $0 < t_1^\alpha < x_i < t_2^\alpha < t_3^\alpha < x_{i+1} < t_4^\alpha$, outside an event of vanishing probability as $\varepsilon \to 0$, for all small enough ε, we have that

$$n(\varepsilon^{-1}t_1) \le \zeta_i^{(\varepsilon)} \le n(\varepsilon^{-1}t_2) \le n(\varepsilon^{-1}t_3) \le \zeta_{i+1}^{(\varepsilon)} \le n(\varepsilon^{-1}t_4), \tag{4.39}$$

and thus

$$\tilde{L}\Big(\tilde{X}_{\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}}, n(\varepsilon^{-1}t_3) - n(\varepsilon^{-1}t_2)\Big) \le L\Big(\tilde{X}_{\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}}, \zeta_{i+1}^{(\varepsilon)}\Big) \le \tilde{L}\Big(\tilde{X}_{\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}}, n(\varepsilon^{-1}t_4) - n(\varepsilon^{-1}t_1)\Big), \tag{4.40}$$

where $\tilde{L}(x, n)$ is the number of visits to x up to time n of a random walk with the same jump distribution of X but starting at $\tilde{X}_{\tilde{\varepsilon}^{-1}x_i^{(\varepsilon)}}$. Furthermore, by Lemma 9 above (see Remark 10), outside an event with vanishing probability as $\varepsilon \to 0$, we have equalities on (4.40).

Now the right hand side term of (4.40) (say) has the same distribution as

$$L\big(0, n(\varepsilon^{-1}t_4) - n(\varepsilon^{-1}t_1)\big),$$

and this quantity is bounded above and below by $L(0, an(\varepsilon^{-1}))$ (with a different $a > 0$ for the lower and upper bound) as soon as ε is small enough.

Corollary 8 and (4.23) now imply that for any fixed $a > 0$ the distribution of

$$\bar{r}_{n(\varepsilon^{-1})} \sum_{i=1}^{L(0, an(\varepsilon^{-1}))} T_i^{(1)} \tag{4.41}$$

converges to a mean 1 exponential one as $\varepsilon \to 0$.

From the above we conclude that outside an event with vanishing probability as $\varepsilon \to 0$, given $T > 0$, $\bar{Y}_t^{(\varepsilon,\delta)}$ equals $\bar{Z}_t^{(\varepsilon,\delta)} = \mu^{(\varepsilon)}(\bar{W}_t^{(\varepsilon,\delta)})$ in $[0, T]$, where $\bar{W}^{(\varepsilon,\delta)}$ is the right continuous inverse of

$$\bar{V}_t^{(\varepsilon,\delta)} = \sum_{\substack{i=1,2,\ldots \\ x_i^{(\varepsilon)} \le t^\alpha}} \mu^{(\varepsilon)}(x_i^{(\varepsilon)})\, \bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, \zeta_{i+1}^{(\varepsilon)}) \tag{4.42}$$

with $\{\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, \zeta_{i+1}^{(\varepsilon)})_i,\ i = 1, 2, \ldots\}$ independent for every $\varepsilon > 0$, and $\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, \zeta_{i+1}^{(\varepsilon)})$ converging in distribution as $\varepsilon \to 0$ to a mean 1 exponential random variable for every i. This and (4.29) yield the result.

□

Remark 16 *It also follows from the arguments in the proof of Lemma 15 that* $(\bar{C}_t^{(\varepsilon,\delta)}) \to (\bar{V}_t^{(\delta)})$ *as* $\varepsilon \to 0$ *in distribution on* (D, J_1)*. Since* $\bar{C}_t^{(\varepsilon)} \geq \bar{C}_t^{(\varepsilon,\delta)}$ *and* $\bar{V}_t^{(\delta)} \to \infty$ *as* $t \to \infty$ *almost surely, we have that given* $T, \eta, \delta > 0$*, there exist* $\varepsilon_0, S > 0$ *such that* $\mathbb{P}(\bar{C}_S^{(\varepsilon)} \leq T) \leq \mathbb{P}(\bar{C}_S^{(\varepsilon,\delta)} \leq T) \leq \eta$ *for all* $\varepsilon < \varepsilon_0$*.*

Back to the proof of Lemma 12, it follows from the fact that $\sum_{x \in \mathfrak{T}_\delta^c \cap [0,T]} \mu(x) \to 0$ as $\delta \to 0$ for arbitrary T and from Lemma 15 that

$$(\bar{Z}_t^{(\delta)}) \to (Z_t) \tag{4.43}$$

as $\delta \to 0$ in distribution on (D, J_1).

To conclude, it thus suffices (since J_1 is stronger than d) to argue that

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} d((\bar{Y}_t^{(\varepsilon,\delta)}), (\bar{Y}_t^{(\varepsilon)})) = 0 \tag{4.44}$$

in probability.

To establish (4.44), we let $\bar{x}_0 = 0$ and $\bar{x}_i = (x_i + x_{i+1})/2$, $i \geq 1$, and introduce

$$\check{Y}_t^{(\varepsilon,\delta)} = \mu^{(\varepsilon)}(x_i^{(\varepsilon)}), \text{ if } \bar{C}^{(\varepsilon)}_{\bar{x}_{i-1}^{1/\alpha}} < t \leq \bar{C}^{(\varepsilon)}_{\bar{x}_i^{1/\alpha}},\ i \geq 1. \tag{4.45}$$

Then, (4.44) follows from

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} d((\check{Y}_t^{(\varepsilon,\delta)}), (\bar{Y}_t^{(\varepsilon)})) = 0, \tag{4.46}$$

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} J_1((\bar{Y}_t^{(\varepsilon,\delta)}), (\check{Y}_t^{(\varepsilon,\delta)})) = 0, \tag{4.47}$$

in probability.

To establish (4.46), we claim that it is enough to consider d_T instead of d, with

$$T = T(\varepsilon, S) = \sum_{x \in \bar{\varepsilon}\mathbb{N}} \mu^{(\varepsilon)}(x)\, \bar{T}^{(\varepsilon)}(x, S), \tag{4.48}$$

S fixed arbitrarily. To justify the claim, it suffices to argue that $T \to \infty$ in probability as $S \to \infty$ uniformly in ε at a neighborhood of the origin. This can be done as follows. T clearly dominates $\bar{C}_S^{(\varepsilon,\delta)}$ defined in (4.34) above, and it is the object of Remark 16 to point out that the latter quantity diverges in probability as $S \to \infty$ uniformly in ε around the origin. This closes the argument for the claim. Reasoning now again as in the proof of Lemma 15, we see that outside an event of vanishing probability as $\varepsilon \to 0$, the distance between $\check{Y}^{(\varepsilon,\delta)}$ and $\bar{Y}^{(\varepsilon)}$ is bounded above by

$$\max\{\mu^{(\varepsilon)}(x_i^{(\varepsilon)}),\ i \geq 1,\ x_i^{(\varepsilon)} \leq S^\alpha + 1\} \sum_{x \in \bar{\varepsilon}\mathbb{N} \cap (\mathfrak{T}_\delta^{(\varepsilon)})^c} \mu^{(\varepsilon)}(x)\, \bar{T}^{(\varepsilon)}(x, S), \tag{4.49}$$

and the result follows from (4.32).

To establish (4.47), we again replace D by D_T, this time T deterministic, but otherwise arbitrarily fixed. For an arbitrary $\eta > 0$, let S be as in Remark 16. Then, arguing as in the proof of Lemma 15 (see paragraph around (4.42)), on the event that $\bar{C}_S^{(\varepsilon,\delta)} > T$ and outside an event of vanishing probability as $\varepsilon \to 0$, $(\bar{Y}_t^{(\varepsilon,\delta)})_{t\in[0,T]}$ successively visits the set of states $\{\mu^{(\varepsilon)}(x_i^{(\varepsilon)}),\, i \geq 1 : x_i^{(\varepsilon)} \leq S^\alpha + 1\}$ (not necessarily all of them by time T, but in any case in that order), with respective sojourn times $\{\mu^{(\varepsilon)}(x_i^{(\varepsilon)})\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, \zeta_{i+1}^{(\varepsilon)}),\, i \geq 1 : x_i^{(\varepsilon)} \leq S^\alpha + 1\}$. The same is of course true of $(\check{Y}_t^{(\varepsilon,\delta)})$, except that the sojourn times are given by $\{\bar{C}^{(\varepsilon)}_{\bar{x}_i^{1/\alpha}} - \bar{C}^{(\varepsilon)}_{\bar{x}_{i-1}^{1/\alpha}} =: \check{S}_i^{(\varepsilon)},\, i \geq 1 : x_i^{(\varepsilon)} \leq S^\alpha + 1\}$. Furthermore, for $i \geq 1$ such that $x_i^{(\varepsilon)} \leq S^\alpha + 1$ the difference between $\check{S}_i^{(\varepsilon)}$ and $\mu^{(\varepsilon)}(x_i^{(\varepsilon)})\bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, \zeta_{i+1}^{(\varepsilon)})$ is bounded above by

$$\sum_{x\in\bar{\varepsilon}\mathbb{N}\cap[\hat{x}_i^{1/\alpha},\check{x}_i^{1/\alpha}]\setminus\{x_i^{(\varepsilon)}\}} \mu^{(\varepsilon)}(x)\,\bar{T}^{(\varepsilon)}(x,S), \tag{4.50}$$

with $\hat{x}_i = (3x_{i-1} + x_i)/4$, $\check{x}_i = (x_i + 3x_{i+1})/4$. One way to proceed now is to observe that the expected value of the expression in (4.50) is bounded above by constant times the sum in (4.32) with $t = S$, which vanishes as $\varepsilon \to 0$, and the result follows readily since η is arbitrary. $\square$

Remark 17 *It also follows from the arguments in the last paragraph of the above proof that for all $t > 0$ fixed*

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \mathbb{P}(\bar{Y}_t^{(\varepsilon,\delta)} \neq \check{Y}_t^{(\varepsilon,\delta)}) = 0 \tag{4.51}$$

5 Aging

We will consider the following two aging functions of (Y_t).

$$\bar{R}(s,t) = \mathbb{P}(Y_t = Y_{t+s}), \tag{5.1}$$
$$\bar{\Pi}(s,t) = \mathbb{P}(Y_t = Y_{t+r} \text{ for all } r \in [0,s]), \tag{5.2}$$

with the following result.

Theorem 18 *Under Assumptions A and B, there exist non-trivial functions $R, \Pi : [0,\infty) \to (0,1]$ such that*

$$\lim\nolimits_{t\to\infty} \bar{R}(\theta t, t) = R(\theta), \tag{5.3}$$
$$\lim\nolimits_{t\to\infty} \bar{\Pi}(\theta v_{n(t)}, t) = \Pi(\theta). \tag{5.4}$$

Remark 19 *Let $\hat{X}_t = X_{I_t}$. In the literature one has rather considered the aging functions*

$$R(s,t) = \mathbb{P}(\hat{X}_t = \hat{X}_{t+s}), \tag{5.5}$$
$$\Pi(s,t) = \mathbb{P}(\hat{X}_t = \hat{X}_{t+r} \text{ for all } r \in [0,s]). \tag{5.6}$$

In case τ_0 is a continuous random variable, then we of course have the identities $R(\cdot,\cdot) = \bar{R}(\cdot,\cdot)$ and $\Pi(\cdot,\cdot) = \bar{\Pi}(\cdot,\cdot)$, but not otherwise. In any case, one can show that aging results like (5.3,5.4) hold for $R(\cdot,\cdot)$ and $\Pi(\cdot,\cdot)$ as well, with $R(\cdot)$ and $\Pi(\cdot)$ as limiting aging functions, respectively.

Remark 20 *R and Π turn out to be the same function. We have*

$$R(\theta) = \Pi(\theta) = \frac{\sin(\pi\alpha)}{\pi}\int_{\theta/(1+\theta)}^{1} s^{-\alpha}(1-s)^{\alpha-1}\,ds. \tag{5.7}$$

See (5.19, 5.25) and Remark 22 below.

In order to prove Theorem 18, we will naturally consider the rescaled version of Y with the special strongly converging rescaled environment $\bar{Y}^{(\varepsilon)}$ (see (4.25) above). One ingredient of the proof of (5.3) is a comparison to $\check{Y}^{(\varepsilon,\delta)}$ (see (4.45) above) as follows.

Lemma 21 *For all $t > 0$*

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{P}(\bar{Y}_t^{(\varepsilon)} \neq \check{Y}_t^{(\varepsilon,\delta)}) = 0 \tag{5.8}$$

Proof

We first consider for $T > 0$

$$\int_0^T \mathbb{P}(\bar{Y}_s^{(\varepsilon)} \neq \check{Y}_s^{(\varepsilon,\delta)})\,ds = \mathbb{E}\int_0^T \mathbf{1}\{\bar{Y}_s^{(\varepsilon)} \neq \check{Y}_s^{(\varepsilon,\delta)}\}\,ds. \tag{5.9}$$

Arguing as in the proofs of Lemmas 12 and 15 above, we first fix an arbitrary $\eta > 0$, and then choose S as in Remark 16. Then on the event that $\bar{C}_S^{(\varepsilon)} > T$ and outside an event of vanishing probability as $\varepsilon \to 0$, the integral on the right of (5.9) is bounded above by

$$\max\{\mu^{(\varepsilon)}(x_i^{(\varepsilon)}),\, i \geq 1,\, x_i^{(\varepsilon)} \leq S^\alpha + 1\} \sum_{\substack{x\in\bar{\varepsilon}\mathbb{N}\cap(\mathfrak{T}_\delta^{(\varepsilon)})^c \\ x\leq S^\alpha+1}} \mu^{(\varepsilon)}(x)\,\bar{T}^{(\varepsilon)}(x,S), \tag{5.10}$$

Using now (4.32), we have that the $\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}$ of the expectation of the integral is bounded above by $T\eta$. Since η is arbitrary, we conclude that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_0^T \mathbb{P}(\bar{Y}_s^{(\varepsilon)} \neq \check{Y}_s^{(\varepsilon,\delta)})\,ds = 0. \tag{5.11}$$

We now fix $\delta' > \delta$ and let $I = \min\{i \geq 1 :\, x_i \in \mathfrak{T}_{\delta'}\}$, and define

$$\xi_i^{(\varepsilon)} = \inf\{t \geq 0 :\, \bar{T}^{(\varepsilon)}(x_i^{(\varepsilon)}, t) > 0\},\ i \geq 1. \tag{5.12}$$

We fix $t > 0$ and condition on $\bar{C}_{\xi_{I+1}^{(\varepsilon)}}^{(\varepsilon)} = \mu^{(\varepsilon)}(x_I^{(\varepsilon)})\bar{T}^{(\varepsilon)}(x_I^{(\varepsilon)}, \zeta_{I+1}^{(\varepsilon)}) + \Delta^{(\varepsilon)}$, where $\zeta_\cdot^{(\varepsilon)}$ was defined right above (4.38) above, and $\Delta^{(\varepsilon)}$ is defined by this equality (and (4.30)): it is an absolutely continuous random variable. We now observe that $L\Big(\tilde{X}_{\bar{\varepsilon}^{-1}x_I^{(\varepsilon)}}, \zeta_{I+1}^{(\varepsilon)}\Big)$ is geometrically distributed with success parameter $p^{(\varepsilon)} := \mathbb{P}(\hat{\zeta}_I^{(\varepsilon)} > \zeta_{I+1}^{(\varepsilon)})$, where

$$\hat{\zeta}_I^{(\varepsilon)} = \inf\{n > \zeta_I^{(\varepsilon)} :\, X_n = \bar{\varepsilon}^{-1}x_I^{(\varepsilon)}\}, \tag{5.13}$$

and thus $\bar{T}^{(\varepsilon)}(x_I^{(\varepsilon)}, \xi_{I+1})$ is exponentially distributed with mean $\bar{b}^{(\varepsilon)} := \bar{r}_{n(\varepsilon^{-1})}/p^{(\varepsilon)}$. Now an entirely similar argument to the one justifying the claim in the paragraph of (4.41) above

implies that $\bar{T}^{(\varepsilon)}(x_I^{(\varepsilon)}, \xi_{I+1})$ converges in distribution to a mean 1 exponential random variable as $\varepsilon \to 0$. We must then have that $\bar{b}^{(\varepsilon)} \to 1$ as $\varepsilon \to 0$.

We thus have that

$$\mathbb{P}(\bar{Y}_t^{(\varepsilon)} \neq \check{Y}_t^{(\varepsilon,\delta)}) \leq \int_0^t \left(\int_0^s c_\varepsilon e^{-c_\varepsilon(s-r)} f^{(\varepsilon)}(r)\, dr \right) \mathbb{P}(\bar{Y}_t^{(\varepsilon)} \neq \check{Y}_t^{(\varepsilon,\delta)} | \bar{C}^{(\varepsilon)}_{\xi_{I+1}^{(\varepsilon)}} = s)\, ds + \mathbb{P}(\bar{C}^{(\varepsilon)}_{\xi_{I+1}^{(\varepsilon)}} \geq t), \tag{5.14}$$

where $c_\varepsilon^{-1} = \bar{b}^{(\varepsilon)} \mu^{(\varepsilon)}(x_I^{(\varepsilon)})$ and $f^{(\varepsilon)}$ is the density of $\Delta^{(\varepsilon)}$. The above integral is thus upper bounded by

$$c_\varepsilon \int_0^t \mathbb{P}(\bar{Y}_t^{(\varepsilon)} \neq \check{Y}_t^{(\varepsilon,\delta)} | \bar{C}^{(\varepsilon)}_{\xi_{I+1}^{(\varepsilon)}} = s)\, ds. \tag{5.15}$$

Now the probability inside the latter integral can be written as

$$\mathbb{P}(\ddot{Y}_{t-s}^{(\varepsilon)} \neq \breve{Y}_{t-s}^{(\varepsilon,\delta)}), \tag{5.16}$$

where $(\ddot{Y}_t^{(\varepsilon)})$ and $(\breve{Y}_t^{(\varepsilon,\delta)})$ are defined as $(\bar{Y}_t^{(\varepsilon)})$ and $(\check{Y}_t^{(\varepsilon,\delta)})$ respectively with $(\bar{C}^{(\varepsilon)}_{t+\xi_{I+1}^{(\varepsilon)}})$ replacing $(\bar{C}_t^{(\varepsilon)})$, so we obtain

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \int_0^t \mathbb{P}(\ddot{Y}_{t-s}^{(\varepsilon)} \neq \breve{Y}_{t-s}^{(\varepsilon,\delta)})\, ds = \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \int_0^t \mathbb{P}(\ddot{Y}_s^{(\varepsilon)} \neq \breve{Y}_s^{(\varepsilon,\delta)})\, ds = 0 \tag{5.17}$$

as we did (5.11). Since $\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} c_\varepsilon^{-1} \geq \delta'$, we get that the $\lim_{\delta\to 0} \limsup_{\varepsilon\to 0}$ of the first term in (5.14) vanishes, and since δ' is arbitrary and $\lim_{\delta'\to 0} \limsup_{\varepsilon\to 0} \bar{C}^{(\varepsilon)}_{\xi_{I+1}^{(\varepsilon)}} = 0$ in probability, as can be readily checked, the result follows. $\square$

Proof of Theorem 18

We will first replace $\bar{Y}^{(\varepsilon)}$ by $\bar{Y}^{(\varepsilon,\delta)}$ and then resort to Lemma 21 and Remark 17. By Lemma 15, we have that

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} (\bar{Y}_t^{(\varepsilon,\delta)}) = (Z_t) \tag{5.18}$$

in distribution on (D, J_1). From that and Lemma 21 and Remark 17 above, we claim that

$$\lim_{\varepsilon\to 0} \bar{R}(\theta\varepsilon^{-1}, \varepsilon^{-1}) = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \mathbb{P}(\bar{Y}_1^{(\varepsilon,\delta)} = \bar{Y}_{1+\theta}^{(\varepsilon,\delta)}) = \mathbb{P}(Z_1 = Z_{1+\theta}) := R(\theta). \tag{5.19}$$

(5.3) then follows. The only point of the claim that needs argueing is the second equality. We first point out that, from the construction of $\bar{Y}^{(\varepsilon,\delta)}$ (see (4.35)) and, since the $\mu(x_i)$'s are almost surely all distinct, from (4.29), for all fixed δ and all small enough ε, the probability in the second term in (5.19) equals

$$\mathbb{P}(\bar{Y}_1^{(\varepsilon,\delta)} = \bar{Y}_{1+r}^{(\varepsilon,\delta)} \text{ for all } r \in [0,\theta]) \tag{5.20}$$

plus a small error, and the latter probability equals

$$\mathbb{P}([1, 1+\theta] \cap \text{range of } \bar{C}^{(\varepsilon,\delta)} = \emptyset). \tag{5.21}$$

It readily follows from that and Remarks 14 and 16 that the second term in (5.19) equals

$$\mathbb{P}([1, 1+\theta] \cap \text{range of } V = \emptyset) = \mathbb{P}(Z_1 = Z_{1+r} \text{ for all } r \in [0,\theta]). \tag{5.22}$$

Now the right hand side of (5.22) equals that of (5.19) (since the $\mu(x_i)$'s are almost surely all distinct).

In the above proof, we felt the need to go through (5.20) and (5.21), since the (indicators of the) events in the the second probability in (5.19) and in (5.20) are not almost surely continuous on (D, J_1), but so is the event in (5.21).

As regards (5.4), let

$$\hat{\Pi}^{(\varepsilon)}(\theta) := \mathbb{P}(\bar{Y}_1^{(\varepsilon)} = \bar{Y}_{1+r}^{(\varepsilon)} \text{ for all } r \in [0, \bar{r}_{n(\varepsilon^{-1})}\theta]). \tag{5.23}$$

One now checks that

$$\hat{\Pi}^{(\varepsilon)}(\theta) = \mathbb{E}\left(e^{-\theta/\bar{Y}_1^{(\varepsilon)}}\right), \tag{5.24}$$

and Lemma 21 implies that

$$\lim_{\varepsilon\to0}\mathbb{E}\left(e^{-\theta/\bar{Y}_1^{(\varepsilon)}}\right) = \lim_{\delta\to0}\lim_{\varepsilon\to0}\mathbb{E}\left(e^{-\theta/\check{Y}_1^{(\varepsilon,\delta)}}\right) = \mathbb{E}\left(e^{-\theta/Z_1}\right) =: \Pi(\theta), \tag{5.25}$$

where the second equality follows from (5.18), which implies marginal convergence in distribution (since each fixed deterministic time is almost surely a continuity point of Z). $\square$

Remark 22 *One quickly checks that*

$$\mathbb{P}(Z_1 = Z_{1+\theta}) = \mathbb{P}([1, 1+\theta] \cap \text{ range of } V = \emptyset) \tag{5.26}$$

(see (2.11, 2.12)). One can then obtain

$$\frac{\sin(\pi\alpha)}{\pi}\int_{\theta/(1+\theta)}^{1} s^{-\alpha}(1-s)^{\alpha-1}\, ds \tag{5.27}$$

as an expresssion for the latter probability. (This may be readily seen to follow from Proposition 3.1 in [11].) We further notice that we can write

$$\mathbb{P}(Z_1 = Z_{1+\theta}) = \mathbb{E}\left(e^{-\theta/Z_1}\right). \tag{5.28}$$

See Remark 20 above. (5.26) and (5.28) give us the Laplace transform of $1/Z_1$. An expression for the density of that variable can be found in (5.97) of [19].

Remark 23 *Another aging function which is natural on one side and not as considered in the literature as the above ones on the other side, and also fits well in the above picture, is the following one.*

$$\Omega(s,t) = \mathbb{P}(\sup_{r\in[0,t]} Y_r < \sup_{r\in[0,t+s]} Y_r) \tag{5.29}$$

It was suggested in [17] as a "measure of the prospects for novelty in the system". Since $\sup_{r\in[0,t]} \bar{Y}_r^{(\varepsilon)} = \sup_{r\in[0,t]} \check{Y}_r^{(\varepsilon,\delta)}$ if $\bar{Y}_t^{(\varepsilon)} = \check{Y}_t^{(\varepsilon,\delta)}$, from Lemma 21, (4.47), Lemma 15 and (4.43), we have that

$$\lim_{t\to\infty}\Omega(\theta t, t) = \mathbb{P}(\sup_{r\in[0,1]} Z_r < \sup_{r\in[0,1+\theta]} Z_r) =: \Omega(\theta). \tag{5.30}$$

This is an example where the limiting aging function requires full use of the process Z; in the previous ones, the limits could be expressed in terms of the (clock) process V alone. We could not find an explicit expression for the right hand side of (5.30).

6 Stronger convergence

In this section, we strengthen the convergence results of Section 4 under an additional condition, which we now explain.

Let $X' = (X'_n)_{n\geq 0}$ a random walk independent from and equally distributed with X and define

$$\mathcal{I}_n = \mathcal{I}_n(X, X') = \{z \in \mathbb{Z}^d : X_i = X'_j = z \text{ for some } 0 \leq i, j \leq n\} = \mathcal{R}_n(X) \cap \mathcal{R}_n(X'),\ n \geq 0, \tag{6.1}$$

as the set of intersection points of the paths of X and X' up to (discrete) time n (it can be seen also as indicated as the intersection of the ranges of X and X' up to time n). Let now

$$I_n = |\mathcal{I}_n| \tag{6.2}$$

be the number of such intersection points. The additional condition we impose, in order that the results of this section hold, is as follows.

Assumption C

$$\frac{I_n}{\mathbb{E}(R_n)} \to 0 \text{ in probability as } n \to \infty. \tag{6.3}$$

Remark 24 *The expectation of the quotient in (6.3) can be reexpressed as*

$$\frac{\sum_{x\in\mathbb{Z}^d}[\mathbb{P}(T_x \leq n)]^2}{\sum_{x\in\mathbb{Z}^d}\mathbb{P}(T_x \leq n)}, \tag{6.4}$$

where $T_x = \inf\{n \geq 0 : X_n = x\}$. We readily find that (6.3) holds in either the general $d \geq 2$ transient case, or the one dimensional non integrable increment, transient case, since in both these cases $\limsup_{\|x\|\to\infty} \mathbb{P}(T_x < \infty) = 0$ (see e.g. Proposition 25.3 in [26]), and in general $\lim_{n\to\infty} \mathbb{E}(R_n) = \infty$.

It also holds for two dimensional mean zero, finite second moment random walks from results in [22]. We are uncertain about other recurrent planar walks, as well as 1-stable 1-dimensional recurrent walks. (See Remark 2 above.)

Let $\mathcal{B}_u$ be the class of bounded uniformly continuous real functions on (D, d). Here is the main result of this section. Let $Y^{(\varepsilon)}$ and Z be as in Theorem 5 above.

Theorem 25 *Under Assumptions A, B and C, for every $F \in \mathcal{B}_u$, we have*

$$\mathbb{E}\left[F(Y^{(\varepsilon)})\middle|\,\tau\right] \to \mathbb{E}\left[F(Z)\right], \tag{6.5}$$

in probability as $\varepsilon \to 0$.

Remark 26 *As anticipated at the end of Section 2, a condition like (6.3) is needed for the validity of the above result. A case where Assumptions A and B are satisfied, but not Assumption C, and (6.5) does not hold, is when X is one dimensional simple asymmetric. This is particularly clear in the totally asymmetric case, when C_n (see (4.1)) is a partial sum of i.i.d. random variables in the basin of attraction of an α-stable law, $\alpha \in (0, 1)$, in which case it is well known to only converge, when properly rescaled, in law. This prevents a result of the form of (6.5) in that case.*

Proof of Theorem 25

Let $F \in \mathcal{B}_u$ be fixed. We may and will restrict to F with bounded support, say $[0, T]$, where $T > 0$ is arbitrary.

It follows from Theorem 5 that

$$\mathbb{E}\left[F(Y^{(\varepsilon)})\right] \to \mathbb{E}\left[F(Z)\right]. \tag{6.6}$$

We will use this and (6.3) to get (6.5).

Let $X^{(1)}, X^{(2)}, \ldots$ and $\tau^{(0)}, \tau^{(1)}, \ldots$ be iid copies of X and τ respectively.

For $k \geq 1$, let $\mathcal{R}^{(k)}$ be defined as in (2.7), with $X^{(k)}$ replacing X.

Let now $\mathcal{Z}_n^{(1)} = \mathcal{R}_n^{(1)}$ and for $k > 1$

$$\mathcal{Z}_n^{(k)} = \mathcal{R}_n^{(k)} \setminus \left\{\cup_{i=1}^{k-1} \mathcal{R}_n^{(i)}\right\}. \tag{6.7}$$

We can then define for each $N \geq 1$

$$\tilde{\tau}^{(N)} = \{\tilde{\tau}_x^{(N)},\, x \in \mathbb{Z}^d\}, \tag{6.8}$$

where $\tilde{\tau}_x^{(N)} = \tau_x^{(k)}$, if $x \in \mathcal{Z}_N^{(k)}$ for some $k \geq 1$, and $\tilde{\tau}_x^{(N)} = \tau_x^{(0)}$, otherwise.

Remark 27 *$\tilde{\tau}^{(N)}$ and τ are equally distributed for every $N \geq 1$, whether or not $X^{(1)}, X^{(2)}, \ldots$ are given. In particular, $\tilde{\tau}^{(N)}$ is independent of $X^{(1)}, X^{(2)}, \ldots$.*

Now let us consider two classes of random functions $C^{(k,N)}, C^{(k)} : \mathbb{N} \to [0, \infty)$, $k, N \geq 1$:

$$C_n^{(k,N)} = \sum_{i=0}^{n} \tilde{\tau}^{(N)}_{X_i^{(k)}} T_i^{(k)}, \quad C_n^{(k)} = \sum_{i=0}^{n} \tau^{(k)}_{X_i^{(k)}} T_i^{(k)}, \quad n \geq 0, \tag{6.9}$$

where $\{T_i^{(k)},\, i \geq 1\} =: T^{(k)}$, $k \geq 1$, are independent families of iid mean 1 exponentials, and their respective right continuous inverses $I^{(k,N)}$ and $I^{(k)}$, and for $t \geq 0$. Let then

$$Y_t^{(k,N)} = \tilde{\tau}^{(N)}_{X^{(k)}_{I_t^{(k,N)}}}, \quad Y_t^{(k)} = \tau^{(k)}_{X^{(k)}_{I_t^{(k)}}}. \tag{6.10}$$

Remark 28 *We remark that $C^{(k,N)} = C^{(k)} = C$ and $Y^{(k,N)} = Y^{(k)} = Y$ in distribution for all k, N; and that $Y^{(k)}$, $k \geq 1$, are iid; and $Y^{(k,N)}$, $k \geq 1$, are iid given $\tilde{\tau}^{(N)}$ for all N. The latter fact follows from the independence of $\tilde{\tau}^{(N)}$ from $X^{(1)}, X^{(2)}, \ldots$ as remarked above (see Remark 27).*

Let us now fix $\varepsilon, \delta > 0$ and define for each $k, N \geq 1$

$$\tau^{(k,\varepsilon,\delta)} = \{\tau_x^{(k,\varepsilon,\delta)} := \tau_x^{(k)} 1\{\tau_x^{(k)} > \delta a_\varepsilon^{-1}\},\, x \in \mathbb{Z}^d\}, \tag{6.11}$$

$$\tilde{\tau}^{(N,\varepsilon,\delta)} = \{\tilde{\tau}_x^{(N,\varepsilon,\delta)} := \tilde{\tau}_x^{(N)} 1\{\tilde{\tau}_x^{(N)} > \delta a_\varepsilon^{-1}\},\, x \in \mathbb{Z}^d\}, \tag{6.12}$$

and let

$$C^{(k,N,\varepsilon,\delta)} = \sum_{i=0}^{n} \tilde{\tau}^{(N,\varepsilon,\delta)}_{X_i^{(k)}} T_i^{(k)}, \quad C^{(k,\varepsilon,\delta)} = \sum_{i=0}^{n} \tau^{(k,\varepsilon,\delta)}_{X_i^{(k)}} T_i^{(k)}, \quad n \geq 0, \tag{6.13}$$

with $I^{(k,N,\varepsilon,\delta)}$ and $I^{(k,\varepsilon,\delta)}$ their respective inverses, and

$$Y_t^{(k,N,\varepsilon)} = a_\varepsilon \tilde{\tau}^{(N)}_{X^{(k)}_{I^{(k,N)}_{\varepsilon^{-1}t}}}, \quad Y_t^{(k,N,\varepsilon,\delta)} = a_\varepsilon \tilde{\tau}^{(N)}_{X^{(k)}_{I^{(k,N,\varepsilon,\delta)}_{\varepsilon^{-1}t}}}, \quad Y_t^{(k,\varepsilon)} = a_\varepsilon \tau^{(k)}_{X^{(k)}_{I^{(k)}_{\varepsilon^{-1}t}}}, \quad Y_t^{(k,\varepsilon,\delta)} = a_\varepsilon \tau^{(k)}_{X^{(k)}_{I^{(k,\varepsilon,\delta)}_{\varepsilon^{-1}t}}}. \tag{6.14}$$

We then have that $Y^{(k,N,\varepsilon)} = Y^{(k,\varepsilon)} = Y^{(\varepsilon)}$ in distribution; $Y^{(k,\varepsilon)}$, $k \geq 1$, are iid; and $Y^{(k,N,\varepsilon)}$, $k \geq 1$, are iid given $\tilde{\tau}^{(N)}$ for all N.

Consider now

$$\frac{1}{K}\sum_{k=1}^{K} F(Y^{(k,N,\varepsilon)}) = \frac{1}{K}\sum_{k=1}^{K} F(Y^{(k,N,\varepsilon,\delta)}) + \frac{1}{K}\sum_{k=1}^{K} \Delta^{(k,N,\varepsilon,\delta)}, \tag{6.15}$$

where K is an arbitrary positive integer, and $\Delta^{(k,N,\varepsilon,\delta)} = F(Y^{(k,N,\varepsilon)}) - F(Y^{(k,N,\varepsilon,\delta)})$.

With an argument similar to the one giving (4.44), one finds that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \Delta^{(k,N,\varepsilon,\delta)} = 0 \tag{6.16}$$

in probability for all k.

Since $C^{(k,N)} = C^{(k)} = C$ in distribution, we have that, given $\eta > 0$, there exists $S = S_K > 0$ such that

$$\mathbb{P}\left(C^{(k,N)}_{n(\varepsilon^{-1}S)} \leq \varepsilon^{-1}T\right) = \mathbb{P}\left(C^{(k)}_{n(\varepsilon^{-1}S)} \leq \varepsilon^{-1}T\right) \leq \frac{\eta}{2K} \tag{6.17}$$

for all $N \geq 1$ and ε sufficiently small (see Remark 16 above).

From now on we take $N = N_\varepsilon = 2\rho_{n(\varepsilon^{-1}S)}$.

For $k, \ell = 1, \ldots, K$, let

$$A_{k,\ell} = \{\tau^{(k)}_{X_i^{(k)}} > \delta a_\varepsilon^{-1} \text{ and } X_i^{(k)} \in \mathcal{R}^{(\ell)}_{n(\varepsilon^{-1}S)} \text{ for some } i \leq n(\varepsilon^{-1}S)\}, \quad A_K = \cup_{k,\ell=1}^{K} A_{k,\ell}. \tag{6.18}$$

Let also

$$\mathcal{I}_{k,\ell} := \mathcal{I}_{n(\varepsilon^{-1}S)}(X^{(k)}, X^{(\ell)}) \tag{6.19}$$

(see (6.1)).

Then, given $\xi > 0$

$$\mathbb{P}(A_{k,\ell}) = \mathbb{P}\left(\sum_{x\in\mathcal{I}_{k,\ell}} 1\{\tau_x^{(k)} > \delta a_\varepsilon^{-1}\} \geq 1\right) \leq \xi N \mathbb{P}(\tau_0 > \delta a_\varepsilon^{-1}) + \mathbb{P}(|\mathcal{I}_{k,\ell}| > \xi N) + \mathbb{P}(R^{(k)}_{n(\varepsilon^{-1}S)} > N). \tag{6.20}$$

By Assumptions A and B, (2.16, 4.12, 4.14, 2.19), the first term on the right of (6.20) is bounded above by

$$3\delta^{-\alpha} S^\alpha \xi m \mathbb{P}(\tau_0 > s_m) \tag{6.21}$$

for all ε small enough, where $m = n(\varepsilon^{-1})$. Using Assumptions A and B and (2.16) again, we may replace $m\mathbb{P}(\tau_0 > s_m)$ by 1 in (6.21).

Putting this, Assumption A and (6.3) together, we conclude that $\mathbb{P}(A_{k,\ell}) \to 0$ as $\varepsilon \to 0$ for all k, ℓ, and thus

$$\mathbb{P}(A_K) \to 0 \tag{6.22}$$

as $\varepsilon \to 0$ for every $K \geq 1$.

We now go back to (6.15). By the above, the first term on its right can be written as

$$\frac{1}{K}\sum_{k=1}^{K} F(Y^{(k,\varepsilon)}) + \frac{1}{K}\sum_{k=1}^{K} \Delta^{(k,\varepsilon,\delta)} + 1\{A_K \cup B_K\}\frac{1}{K}\sum_{k=1}^{K}(F(Y^{(k,N,\varepsilon,\delta)}) - F(Y^{(k,\varepsilon,\delta)})), \qquad (6.23)$$

where $\Delta^{(k,\varepsilon,\delta)} = F(Y^{(k,\varepsilon,\delta)}) - F(Y^{(k,\varepsilon)})$ and $B_K = \bigcup_{k=1}^{K}\Big\{\{C^{(k,N)}_{n(\varepsilon^{-1}S)} \leq \varepsilon^{-1}T\} \cup \{C^{(k)}_{n(\varepsilon^{-1}S)} \leq \varepsilon^{-1}T\}\Big\}$. This follows from the fact that outside $A_K \cup B_K$, we have that $Y_t^{(k,\varepsilon,\delta)} = Y_t^{(k,N,\varepsilon,\delta)}$ for $t \in [0,T]$ and $1 \leq k \leq K$, and all fixed ε, δ, N, as can be readily checked.

As in (6.16), we have that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \Delta^{(k,\varepsilon,\delta)} = 0 \qquad (6.24)$$

in probability for all k.

Now from (6.15,6.23), and since $F \in \mathcal{B}_u$, we get

$$\left|\mathbb{E}\left[F(Y^{(1,N,\varepsilon)})\middle|\,\tilde\tau^{(N)}\right] - \mathbb{E}\left[F(Z)\right]\right| \leq$$
$$\left|\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\big(F(Y^{(k,N,\varepsilon)}) - \mathbb{E}\big[F(Y^{(k,N,\varepsilon)})\big|\,\tilde\tau^{(N)}\big]\big)\right| + \left|\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\big(F(Y^{(k,\varepsilon)}) - \mathbb{E}\big[F(Y^{(k,\varepsilon)})\big]\big)\right| \qquad (6.25)$$

plus a term whose expectation is bounded above by

$$\left|\mathbb{E}(F(Y^{(1,N,\varepsilon)})) - \mathbb{E}(F(Z))\right| + \mathbb{E}\left|\Delta^{(1,N,\varepsilon,\delta)}\right| + \mathbb{E}\left|\Delta^{(1,\varepsilon,\delta)}\right| + 2\|F\|_\infty\left(\mathbb{P}(A_K) + \mathbb{P}(B_K)\right), \qquad (6.26)$$

where we have used the fact that both $\mathbb{E}\big[F(Y^{(k,\varepsilon)})\big]$ and $\mathbb{E}\big[F(Y^{(k,N,\varepsilon)})\big|\,\tilde\tau^{(N)}\big]$ are independent of k.

Recalling now Remark 28, and using Jensen, we find that the right hand side of (6.25) is bounded above by constant times $K^{-1/2}$. This and (6.6, 6.16, 6.17, 6.22, 6.24) yield

$$\limsup_{\varepsilon\to 0}\mathbb{E}\left|\mathbb{E}\left[F(Y^{(1,N,\varepsilon)})\middle|\,\tilde\tau^{(N)}\right] - \mathbb{E}\left[F(Z)\right]\right| \leq \mathrm{const}\,(K^{-1/2} + \eta). \qquad (6.27)$$

Since K and η are arbitrary and the left hand side of (6.27) does not depend on either, we conclude that

$$\mathbb{E}\left[F(Y^{(1,N,\varepsilon)})\middle|\,\tilde\tau^{(N)}\right] \to \mathbb{E}\left[F(Z)\right] \qquad (6.28)$$

in probability as $\varepsilon \to 0$, and the result follows from the fact that $\mathbb{E}\left[F(Y^{(1,N,\varepsilon)})\middle|\,\tilde\tau^{(N)}\right]$ and $\mathbb{E}\left[F(Y^{(\varepsilon)})\middle|\,\tau\right]$ have the same distribution for every $\varepsilon > 0$, $N \geq 1$. $\square$

6.1 Stronger aging results

The above can be extended to strengthen the aging results of the previous section, under the same conditions of this section (namely (6.3)).

Theorem 29 *Under Assumptions A, B and C, we have that*

$$\lim_{t\to\infty}\mathbb{P}(Y_t = Y_{t+\theta t}|\tau) = R(\theta), \qquad (6.29)$$
$$\lim_{t\to\infty}\mathbb{P}(Y_t = Y_{t+r} \text{ for all } r \in [0, \theta v_{n(t)}]|\tau) = \Pi(\theta), \qquad (6.30)$$

in probability as $t \to \infty$, where R and Π are as in Theorem 18 (and indeed, both equal the right hand side of (5.26) above).

Sketch of proof

An argument like that of Lemma 15 can be used to get that

$$(Y_t^{(1,\varepsilon,\delta)}) \to (\bar{Z}_t^{(\delta)}) \tag{6.31}$$

as $\varepsilon \to 0$ in distribution on (D, J_1) (in here, differently from Lemma 15 case, τ is integrated; the argument might of course use a version of τ such that (6.31) holds for τ in a set of full probability). We may then extend Theorem 25 with a similar proof to get

$$\mathbb{E}\left[G(Y^{(1,\varepsilon,\delta)})\middle|\,\tau\right] \to \mathbb{E}\left[G(\bar{Z}^{(\delta)})\right], \tag{6.32}$$

in probability as $\varepsilon \to 0$, for every $\delta > 0$, with $G : D \in \{0, 1\}$ such that either

$$G(U) = 1\{U_1 = U_{1+r} \text{ for all } r \in [0, \theta]\} \text{ or } G(U) = 1\{U_1 = U_{1+\theta}\},\ U \in D. \tag{6.33}$$

We further need to extend Lemma 21 to get that for all $t > 0$

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}(Y_t^{(1,\varepsilon)} \neq Y_t^{(1,\varepsilon,\delta)}) = 0 \tag{6.34}$$

(here the proof can be made again by using special versions of τ like in the argument for Lemma 21; it is at this point that the stronger Assumption B' is used, like in the proof of that lemma).

From (6.32) and (6.34), and since $\mathbb{E}\left[G(\bar{Z}^{(\delta)})\right] \to \mathbb{E}\left[G(Z)\right]$ as $\delta \to 0$, we get

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[G(Y^{(1,\varepsilon)})\middle|\,\tau\right] = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \mathbb{E}\left[G(Y^{(1,\varepsilon,\delta)})\middle|\,\tau\right] = \lim_{\delta\to 0} \mathbb{E}\left[G(\bar{Z}^{(\delta)})\middle|\,\tau\right] = \mathbb{E}\left[G(Z)\right], \tag{6.35}$$

in probability as $\varepsilon \to 0$, for G as in (6.33). □

Remark 30 *Under the same conditions of Theorem 29, and by the same reasoning as above, we have that*

$$\lim_{t\to\infty} \mathbb{P}(\sup_{r\in[0,t]} Y_r < \sup_{r\in[0,t+\theta t]} Y_r|\tau) = \Omega(\theta) \tag{6.36}$$

in probability. (See Remark 23 above.)

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